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GENERAL ELECTRIC

SANTA BARBARA, CALIFORNIA

ON AN EXTENSION OF TAYLOR'S BLAST WAVE SOLUTION

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SUMMARY

A solution of the hydrodynamic equations is presented which describes a spherically symmetric shock wave in a polytropic gas resulting from the instantaneous release of an arbitrary amount of energy in the center. The external force of gravity as well as some variable density and pressure distribution of the undisturbed medium are taken into account. It is shown that the head of the outwards moving shock wave is located at a distance $R(t) = \sigma \cdot t^{2/3}$ from the center. The constant σ is determined from the adiabatic exponent of the gas, the density and pressure coefficients of the ambient medium and the amount of energy supplied by the explosion.

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INTRODUCTION

Sir G. I. Taylor [1] found a solution of the hydrodynamic equations which represents a spherically symmetric strong-shock wave in a polytropic gas resulting from a very intense explosion in the center. He neglected the gravitational effect and assumed further that density and pressure of the ambient medium are constant and zero, respectively. The latter assumption restricts the solution automatically to the strong-shock regime.

Other strong-shock solutions of Taylor's type have been found with regard to the geometry of the problem [2], and with regard to the inclusion of an additional heat flux term in the hydrodynamic equations [3]. The spherically symmetric blast wave problem including the gravitational effect and some variable density distribution of the ambient medium has been solved for a strong shock in [4].

The purpose of the present investigation is to provide a spherically symmetric solution of Taylor's type for an arbitrary (strong or weak) shock under consideration of the gravitational effect and some density and pressure distribution of the ambient gas as functions of the distance from the center. This solution may be applied, for example, to the description of the hydrodynamic phenomenae resulting from the birth of a nova.

ANALYTIC FORMULATIONS AND ASSUMPTIONS

The problem is to determine radial particle velocity u, pressure p and density ρ within an outwards moving spherically symmetric shock wave as functions of radial distance r from the center and time t. In the case of a polytropic gas with the adiabatic constant γ, these quantities have to satisfy the hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 \rho u)}{\partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{\partial \Phi}{\partial r} \qquad (1)$$

$$\frac{\partial (p\rho^{-\gamma})}{\partial t} + u\frac{\partial (p\rho^{-\gamma})}{\partial r} = 0$$

where $\Phi(r)$ is the gravitational potential per unit mass. A further unknown in the problem is the position $r = R(t)$ of the shock front at time t after the explosion occurred.

The solutions u, p and ρ of the hydrodynamic equations (1) are subject to the Rankine-Hugoniot transition conditions at the head of the shock wave. Let u_1, p_1 and ρ_1 denote the values of u, p and ρ immediately behind the shock front, and u_0, p_0 and ρ_0 those immediately ahead of the shock front. The Rankine-Hugoniot relations are

$$\rho_1(\dot{R} - u_1) - \rho_0(\dot{R} - u_0) = 0$$

$$\rho_1 u_1(\dot{R} - u_1) - \rho_0 u_0(\dot{R} - u_0) = p_1 - p_0 \qquad (2)$$

$$\rho_1\left(\frac{1}{2}u_1^2 + \frac{1}{\gamma-1}\frac{p_1}{\rho_1}\right)(\dot{R} - u_1) - \rho_0\left(\frac{1}{2}u_0^2 + \frac{1}{\gamma-1}\frac{p_0}{\rho_0}\right)(\dot{R} - u_0)$$

$$= p_1 u_1 - p_0 u_0$$

Here, $\dot{R}$ stands for the velocity of the shock front.

The second restriction on u, p, and ρ is that the amount of energy supplied by the (instantaneous) explosion is to be constant with respect to time after the burst. If $u_0^*(r)$, $p_0^*(r)$ and $\rho_0^*(r)$ denote the particle velocity, pressure and density of the gas in the (stationary) case prior to the explosion, this amount of energy is given by

$$E = 4\pi \int_0^{R(t)} \left[\frac{1}{2} (\rho u^2 - \rho_0^* u_0^{*2}) - \Phi \cdot (\rho - \rho_0^*) + \frac{1}{\gamma - 1} (p - p_0^*) \right] r^2 dr \tag{3}$$

The second requirement can be expressed by the equation

$$\frac{dE}{dt} = 0, \quad t > t_0 = 0 \tag{4}$$

where $t = t_0$ denotes the instant when the explosion occurs.

We assume the functions $u_0^*(r)$, $p_0^*(r)$, $\rho_0^*(r)$ and the gravitational potential $\Phi(r)$ to be of the form

$$\begin{aligned} u^*(r) &= 0 \\ p^*(r) &= \frac{\lambda}{r^3} \\ \rho^*(r) &= \frac{\mu}{r^2} \\ \Phi(r) &= -\frac{3\lambda}{\mu} \cdot \frac{1}{r} \end{aligned} \tag{5}$$

Here λ and μ are positive constants. Note that these functions do solve the hydrodynamic equations in the stationary (or more precisely: in the quiescent) case.

For $r = R$, the quantities $u_0^*(R)$, $p_0^*(R)$ and $\rho_0^*(R)$ are identical with u_0, p_0 and ρ_0, respectively. The Rankine-Hugoniot relations (2), therefore, can be written as

$$u_1 = \frac{2}{(\gamma+1)\dot{R}} \left[\dot{R}^2 - \nu \frac{\lambda}{\mu} \cdot \frac{1}{R} \right]$$

$$p_1 = \frac{\mu}{(\gamma+1)R^2} \left[2\dot{R}^2 - (\gamma-1)\frac{\lambda}{\mu} \cdot \frac{1}{R} \right] \tag{6}$$

$$\rho_1 = \frac{\mu(\gamma+1)}{R^2} \left[\frac{\dot{R}^2}{(\gamma-1)\dot{R}^2 + 2\nu \frac{\lambda}{\mu} \cdot \frac{1}{R}} \right]$$

The energy supplied by the explosion is then given by

$$E = 4\pi \int_0^{R(t)} \left[\frac{1}{2}\rho u^2 + \frac{3\lambda}{\mu} \cdot \frac{1}{r}\left(\rho - \frac{\mu}{r^2}\right) + \frac{1}{\gamma-1}\left(p - \frac{\lambda}{r^3}\right) \right] r^2 dr \tag{7}$$

PROGRESSING SHOCK WAVE

Following a classical procedure, we try to solve the present problem by assuming the form of "progressing waves" for the solutions in order to reduce the hydrodynamic equations to a system of ordinary differential equations. These are solutions conveniently written as

$$\begin{aligned} u(r,t) &= a(r) \cdot U(x) \\ p(r,t) &= b(r) \cdot P(x) \\ \rho(r,t) &= c(r) \cdot \Omega(x) \end{aligned} \qquad (8)$$

where x is the combination

$$x(r,t) = \frac{h(r)}{t} \qquad (9)$$

Inserting equations (8) and (9) into the hydrodynamic equations (1), we obtain by straightforward computation the equations

$$\begin{aligned} &\frac{c}{h}\left\{ -x^2\Omega' + \frac{h}{c}\left[(ac)' + \frac{2ac}{r}\right]U\Omega + ah'x\,(U\Omega)'\right\} = 0 \\ &\frac{a}{h}\left\{ -x^2U' + a'hU^2 + ah'xUU' + \frac{b'h}{ac}\cdot\frac{P}{\Omega} + \frac{bh'}{ac}\,x\,\frac{P'}{\Omega}\right\} = -\frac{3\lambda}{\mu}\cdot\frac{1}{r^2} \\ &\frac{bc^{-\gamma}}{h}\left\{ -x^2(P\Omega^{-\gamma})' + U\left[ah\left(\frac{b'}{b} - \gamma\frac{c'}{c}\right)(P\Omega^{-\gamma}) + ah'x(P\Omega^{-\gamma})'\right]\right\} = 0 \end{aligned} \qquad (10)$$

Here a prime denotes the derivative with respect to the argument of the function. Solutions of equations (10) can be obtained by separation of the variables r and x. The separability conditions are

$$
\begin{aligned}
& ah' = \text{const.} \\
& a'h = \text{const.} \\
& \frac{a}{h} = \frac{\text{const.}}{r^2} \\
& \frac{h}{c}\left[(ac)' + \frac{2ac}{r}\right] = \text{const.} \\
& \frac{b'h}{ac} = \text{const.} \\
& \frac{bh'}{ac} = \text{const.} \\
& ah\left[\frac{b'}{b} - \gamma\,\frac{c'}{c}\right] = \text{const.}
\end{aligned}
\tag{11}
$$

The first three equations yield

$$
\begin{aligned}
a(r) &= \alpha r^{-1/2} \\
h(r) &= \left(\frac{r}{\sigma}\right)^{3/2}
\end{aligned}
\tag{12a}
$$

while b and c are obtained from the remaining equations as

$$
\begin{aligned}
b(r) &= \beta r^{\omega - 1/2} \\
c(r) &= \delta r^{\omega + 1/2}
\end{aligned}
\tag{12b}
$$

α, β, δ, σ and ω are constants.

The position of the shock can be determined by setting

$$
x = \frac{h(r)}{t} = \left(\frac{r}{\sigma}\right)^{3/2} \cdot \frac{1}{t} = 1 \quad \text{or} \quad r = R(t) = \sigma t^{2/3}
\tag{13}
$$

Without limitation of generality we can choose

$$\alpha = 1$$
$$\beta = \lambda \tag{14}$$
$$\delta = \mu$$

The energy supplied by the explosion is then given by

$$E = 4\pi \int_0^{R(t)} \Big[\frac{1}{2} \mu r^{\omega-1/2} \Omega(x) U^2(x) + 3\lambda (r^{\omega-1/2} \Omega(x) - r^{-3}) + \frac{\lambda}{\gamma-1} (r^{\omega-1/2} P(x) - r^{-3}) \Big] r^2 dr \tag{15}$$

In order that the energy be independent of time, we set

$$\omega = -\frac{5}{2} \tag{16}$$

Introducing x as the integration variable for arbitrary but fixed time t, one obtains

$$E = \frac{8\pi}{3} \int_0^1 \Big[\frac{1}{2} \mu \Omega(x) U^2(x) + 3\lambda(\Omega(x) - 1) + \frac{\lambda}{\gamma-1} (P(x) - 1) \Big] \frac{1}{x} \, dx \tag{17}$$

The solution for the shock wave is represented by the functions

$$u(r,t) = r^{-1/2} U(x)$$
$$p(r,t) = \lambda r^{-3} P(x)$$
$$\rho(r,t) = \mu r^{-2} \Omega(x) \tag{18}$$
$$x = \left(\frac{r}{\sigma}\right)^{3/2} \cdot \frac{1}{t}$$

where U(x), P(x) and $\Omega(x)$ have to be determined from the system of first order ordinary differential equations

$$(2x^2 - 3\epsilon x U)\Omega' + \epsilon(U - 3xU')\Omega = 0$$

$$\left[\mu(2x^2 - 3\epsilon x U)U' + \epsilon(\mu U^2 - 6\lambda)\right]\Omega + 3\lambda\epsilon(2P - xP') = 0 \qquad (19)$$

$$(2x^2 - 3\epsilon x U)(P\Omega^{-\gamma})' + 2\epsilon(2\gamma - 3)UP\Omega^{-\gamma} = 0$$

with

$$\epsilon = \sigma^{-3/2} \qquad (20)$$

The initial conditions for $U(x)$, $P(x)$ and $\Omega(x)$ are obtained from the Rankine-Hugoniot relations at the shock front $x = 1$. First one derives from equation (13) the velocity for the shock front

$$\dot{R} = \frac{2}{3}\sigma t^{-1/3} = \frac{2}{3\epsilon} R^{-1/2} \qquad (21)$$

Equations (6) become

$$u_1 = \left[\frac{4\mu - 9\nu\lambda\epsilon^2}{3\mu(\gamma+1)\epsilon}\right] R^{-1/2}$$

$$p_1 = \left[\frac{8\mu - 9(\gamma-1)\lambda\epsilon^2}{9(\gamma+1)\epsilon^2}\right] R^{-3} \qquad (22)$$

$$\rho_1 = \left[\frac{2\mu^2(\gamma+1)}{2\mu(\nu-1)+9\gamma\lambda\epsilon^2}\right] R^{-2}$$

On the other hand, the same values for u_1, p_1 and ρ_1 are obtained from equations (18) if we set $r = R$ and $x = 1$. The desired initial conditions are therefore

$$U(1) = \frac{4\mu - 9\gamma\lambda\epsilon^2}{3\mu(\gamma+1)\epsilon} \qquad (23)$$

$$
\begin{aligned}
P(1) &= \frac{8\mu - 9(\gamma-1)\lambda\varepsilon^2}{9(\gamma+1)\lambda\varepsilon} \\
\Omega(1) &= \frac{2\mu(\gamma+1)}{2\mu(\gamma-1)+9\gamma\lambda\varepsilon^2}
\end{aligned}
\tag{23}
$$

The integration of the differential equations (19) can be reduced to the problem of solving a second order ordinary differential equation for $U(x)$ and determining $P(x)$ and $\Omega(x)$ by quadratures. With the abbreviations

$$
\begin{aligned}
\frac{\epsilon(3xU'-U)}{2x^2-3\epsilon xU} &= \Phi \\
\frac{2\epsilon(2\gamma-3)\,U}{2x^2-3\epsilon xU} &= \Psi \\
\frac{\mu}{3\lambda\epsilon}\left[2x^3U' - \epsilon(3xU'-U)\,U\right] - 2 &= \Lambda
\end{aligned}
\tag{24}
$$

it follows from the first and third equation in (19) that

$$
\begin{aligned}
\Omega(x) &= \Omega(1)\, e^{\int_1^x \Phi\, dx} \\
P(x) &= P(1)\, e^{-\int_1^x (\Psi+\gamma\Phi)\, dx}
\end{aligned}
\tag{25}
$$

These explicit expressions for $\Omega(x)$ and $P(x)$ lead to

$$
\frac{xP'-2P}{\Omega} = \frac{P(1)}{\Omega(1)}\left[\Psi(1)+\gamma\Phi(1)-2\right]\cdot e^{\int_1^x \left\{\Psi+(\gamma-1)\Phi\,\frac{\left[x(\Psi+\gamma\Phi)-2\right]'}{x(\Psi+\gamma\Phi)-2}\right\} dx}
\tag{26}
$$

The second equation in (19) reads

$$
\frac{xP'-2P}{\Omega} = \Lambda = \Lambda(1)\, e^{\int_1^x \frac{\Lambda'}{\Lambda}\, dx}
\tag{27}
$$

Equations (26) and (27) furnish

$$\frac{P(1)}{\Omega(1)\Lambda(1)}\left[\Psi(1)+\gamma\Phi(1)-2\right]e^{\int_1^x\left\{\Psi+(\gamma-1)\Phi+\frac{\left[x(\Psi+\gamma\Phi)-2\right]'}{x(\Psi+\gamma\Phi)-2}-\frac{\Lambda'}{\Lambda}\right\}dx} = 1 \tag{28}$$

This equation holds identically in x, if and only if the equations

$$\Psi+(\gamma-1)\Phi+\frac{\left[x(\Psi+\gamma\Phi)-2\right]'}{x(\Psi+\gamma\Phi)-2}-\frac{\Lambda'}{\Lambda} = 0$$

$$\frac{P(1)}{\Omega(1)\Lambda(1)}\left[\Psi(1)+\gamma\Phi(1)-2\right] = 1 \tag{29}$$

are satisfied. Here the first equation represents the second order ordinary differential equation for $U(x)$, while the second equation furnishes the required initial value $U'(1)$ on substitution of (24).

For given physical constants γ, λ and μ, the solutions of equations (25) and (29) will be functions of x and of the parameter ϵ. If $\gamma = 3/2$, the flow is isentropic.

It is meaningful to ask under what conditions the pressure p and the density ρ remain positive. Equations (18) require that $P(x)$ and $\Omega(x)$ be positive. According to equations (25), this is the case if the integrals over Φ and Ψ exist and both constants $\Omega(1)$ and $P(1)$ are positive. It follows then from the second of equations (23) that the parameter ϵ has to be restricted to the interval

$$0 < \epsilon < \frac{2}{3}\sqrt{\frac{2\mu}{3(\gamma-1)\lambda}} \tag{30}$$

Equation (17), after inserting the solutions $U(x, \epsilon)$, $P(x, \epsilon)$ and $\Omega(x, \epsilon)$ and integrating, gives the relation between the energy E supplied by the explosion and the parameter ϵ. If ϵ varies between the limits given in (30), E may range between zero and infinity. Solving equation (17) for ϵ (or σ), one arrives finally at the result that the position of the shock front at time t after the explosion is determined by the distance

$$r = R(t) = \sigma(\gamma, \lambda, \mu, E)\cdot t^{\frac{2}{3}} \tag{31}$$

It is interesting to compare this result with that obtained by Taylor [1] in the case of a strong shock, where density and pressure outside the wave are assumed to be constant and zero, respectively, and gravity is neglected. His result states that the shock front can be localized by the distance $r = R(t) = \sigma^* \cdot t^{2/5}$, where the constant σ^* is determined in a similar manner as σ from the adiabatic constant of the gas, the outside (constant) density and the energy supplied by the explosion.

Comparison of the two results shows that the expansion velocity of the wave in the present case is greater than that of Taylor's wave, at least from a certain time on, depending on the ratio of the constants σ and σ^*. This fact, of course, follows from the different assumptions made; it might be accounted for by the assumed density and pressure distribution only, since the gravity might be expected to cause the opposite effect. Nevertheless, it is remarkable that this is true also if the shock is weak.

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